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A COMBINED MODAL/FINITE ELEMENT ANALYSIS TECHNIQUE FOR THE DYNAMIC RESPONSE OF A NON-LINEAR BEAM TO HARMONIC EXCITATION

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In this paper, a method is proposed for modelling large deflection beam response involving multiple vibration modes. Significant savings in computational time can be obtained compared with the direct integration non-linear finite element method. The deflections from a number of static non-linear finite element test cases are transformed into modal co-ordinates using the modes of the underlying linear system. Regression analysis is then used to find the unknown coupled non-linear modal stiffness coefficients. The inclusion of finite element derived modal masses, and an arbitrary damping model completes the governing non-linear equations of motion. The response of the beam to excitation of an arbitrary nature may then be found using time domain numerical integration of the reduced set of equations. The work presented here extends upon the work of previous researchers to include non-linearly coupled multi-modal response. The particular benefits of this approach are that no linearization is imposed, and that almost any commercial finite element package may be employed without modification.

The proposed method is applied to the case of a homogeneous isotropic beam. Fully simply supported and fully clamped boundary conditions are considered. For the free vibration case, results are compared to those of previous researchers. For the case of steady-state harmonic excitation, results are compared with the direct integration non-linear finite element method using ABAQUS. In all cases, excellent agreement is obtained.

1. INTRODUCTION

The surface panels of modern high-speed aircraft are subjected to high-intensity acoustic loading from sources such as jet efflux and turbulent fluid flow. Frequently this high-intensity noise environment is combined with elevated panel temperatures, caused by aerodynamic heating and jet exhaust impingement. Preliminary concept evaluations of aircraft such as the National Aerospace Plane (NASP) indicate that at some points on the structure, the sound pressure levels will be in the range 170–180 dB (relative to 20 μPa), with panel temperatures up to 1770°C [1]. The issue of acoustic fatigue is also becoming particularly prevalent around the ordinance cavities of high-speed military aircraft, where the acoustic excitation spectrum is a combination of broadband random noise, and high level harmonic "tones" caused by the oscillation of air inside the cavity [2]. The case of an isotropic beam can be regarded as a simplified case of the more complex structural models used in the design of high-speed aircraft components. Because of this, considerable effort has been directed into developing models and solution approaches for the non-linear beam "test bed", which may eventually be applied to more general structures.

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If the transverse displacement response of a beam that is axially constrained at the supports is sufficiently large that the maximum deflection at the center is of the order of the panel thickness, then the inherent non-linear stiffness characteristics of the beam will have a significant effect. The non-linear dynamic behavior may be attributed to the stretching of the beam in the axial direction as it deflects. Simple linear beam theory is no longer applicable in such a situation.

There has been an enormous amount of work published in the study of non-linear beam vibration. It should be noted that only a few of the many authors in this subject are mentioned here. Of the analytical single-degree-of-freedom methods, the accepted benchmark for ideal non-linear simply supported beam behavior is the work of Woinowsky-Krieger [3], which uses the elliptic integral function to evaluate the equation of motion. Both Raju *et al.* [4], and later Lewandowski [5] have applied the Rayleigh–Ritz technique to the non-linear beam vibration problem. The latter work utilizes an iterative approach to determine the natural frequencies and a non-linear mode shape. The work of Rehfield [6, 7] is another notable analytical method for the analysis of non-linear beam vibration. This method is applicable to cases of free vibration, and also forced vibration where the driving excitation is harmonic and spatially arranged so as to only drive a single mode. This method is based around a perturbation procedure.

The work of Rehfield was extended using a semianalytical approach by Ward [8]. Static displacement information derived from a displacement finite element model is utilized in a manner that is paralleled by the present work, but was only applied to a single mode. Another single mode, semianalytical technique is that of Maymon [9]. Rather than using a perturbation technique, this work described a fundamental mode expansion of an arbitrary non-linear structure. A single unknown cubic modal non-linear stiffness coefficient was found by considering the non-linear response to static loading. The stress response was found by treating the stress modes of the structure in a similar manner. The response to random temporal excitation was found using an equivalent linearization scheme. The work presented here extends upon the work of Maymon to include multi-mode vibration in a situation where the form of the non-linear modal couplings is not known *a priori*. A similar use of static finite element displacement data in dynamic analysis has been proposed by Knight [10], although in this instance a multi-mode rather than a single mode approach is adopted. For a consideration of step-impulse loading, a set of reduced basis vectors is formed using the free vibration modes of the underlying linear system, along with a non-linear static solution for the specified step loading under consideration. By contrast the present approach uses only linear free vibration modes as basis vectors, for the consideration of harmonic excitation.

One of the most recent works which employ a similar philosophy to the method presented here is that of Muravyov [11] and Muravyov *et al.* [12]. This approach utilizes a multi-degree-of-freedom modal co-ordinate system. The unknown non-linear stiffness coefficients in modal space are then found by solving a set of prescribed displacement problems using proprietary finite element software. The appropriate choice of prescribed displacement solutions allows the unknown non-linear coefficients to be solved for exactly. By contrast the present approach utilizes a set of prescribed force non-linear static solutions, and a regression analysis. This avoids any potential numerical sensitivity problems that may be associated with using a prescribed displacement solution. In reference [12], two different stochastic linearization schemes are compared for the case of a beam-like problem.

Of the numerically based methods applicable to non-linear vibration analysis, by far the most research effort has been directed towards the development of the finite element method. Sarma and Varadan [13] have used a Lagrangian finite element formulation. Singh *et al.* [14] have also obtained excellent results by using an iterative process to exactly satisfy the axial and out-of-plane equations. Much of the work of Mei and his associates has also utilized the finite element method. Earlier work [15, 16] assumed simple harmonic response behavior and neglected longitudinal displacement and inertia in the finite element formulation. The work of Mei and Decha-Umphai [17] included the effect of transverse displacement and inertia, but simplified the strain displacement relationship by using a linearizing function. The more up-to-date works of Shi and Mei [18], and Shi *et al.* [19] have considered the simply supported beam problem as part of the validation of a formulation developed for laminated composite plates. The finite element equations of motion are expanded in terms of linear modal co-ordinates, then numerically integrated in the time domain. The results show excellent agreement with reference [3] (see Table 2), however, these methods require special finite element codes, or adapted commercial codes.

If the time domain integration technique is used in conjunction with the finite element method for "real" problems, then for computational efficiency a transformation to a reduced basis co-ordinate system is arguably essential [20]. The importance of reducing the system degree of freedom from that of typical finite element models is evidenced by Green and Killey [21], where the CPU time required for non-linear direct integration Monte Carlo analysis of finite element models is prohibitively large.

In this work an alternative approach to that of a "first principles" derivation and solution of the system equations of motion is proposed. The output from a series of static finite element "test cases" is transformed into modal co-ordinates using the mode shapes of the underlying linear system. Regression analysis is then performed in order to extract the non-linear stiffness coefficients in the modal co-ordinate system. Both direct and coupling coefficients may be identified. Time domain numerical integration is then applied to the non-linear modal equations of motion, thus finding the response of the beam to any excitation time history. The method is an approximation, and is "simplified" from the point of view that considerations of the non-linear finite element formulation and solution are handled by the proprietary finite element code, and are not dealt with explicitly by the method. When considering reduced basis methods such as the one proposed here, a number of different forms of reduced basis vectors may be used. In this work, the normal modes of the underlying linear system were chosen as basis vectors. Normal modes were chosen, as the practicing engineer will be more familiar with their use, they may be readily extracted in the appropriate form by using proprietary finite element software, and they may be measured experimentally, together with modal damping information [22].

The proposed method differs from many existing "first principles" formulations in the fact that proprietary finite element packages may be used without modification to the code. An autonomous program post-processes the output from these codes. Significant savings in computational time compared to standard direct integration routines in commercial finite element packages can be obtained without sacrificing the flexibility of the large-scale packages. The derivation that follows is that for an isotropic beam, but it is envisaged that with refinement the method will be expandable to include thermally post-buckled plates and built-up structures such as stiffened panels. The approach will be demonstrated for an initially unstressed beam under free vibration and forced harmonic vibration. Results will be compared with other theories, and with the direct finite element integration technique.

2. FORMULATION

2.1. MODAL TRANSFORMATION

Consider the case of an initially straight, geometrically non-linear beam with mass proportional damping subject to forced vibration. The assembled N degree of freedom finite element equation of motion in physical co-ordinates for forced vibration in the transverse direction is of the form

$$[\mathbf{M}\{\ddot{\mathbf{w}}(x, t\} + \alpha[\mathbf{M}]\dot{\mathbf{w}}\{(x, t)\} + [\mathbf{K}_L]\{\mathbf{w}(x, t)\} + [\mathbf{K}_{NL}(\mathbf{w}(x, t))]\{\mathbf{w}(x, t)\} = \{\mathbf{F}(x, t)\}. \quad (1)$$

Here $\{\mathbf{w}(x, t)\}$ is the transverse deflection vector, $[\mathbf{M}]$ is the assembled mass matrix, α is the Rayleigh coefficient for linear mass proportional damping, $[\mathbf{K}_L]$ is the assembled linear stiffness matrix, $[\mathbf{K}_{NL}]$ is the assembled non-linear stiffness matrix where the stiffness is dependent upon displacement, and $\{\mathbf{F}(x, t)\}$ is the external force vector. The overdots imply differentiation with respect to time. The spatial and temporal components of the beam motion can be separated by expressing the equations of motion in terms of modal amplitudes as

$$\{\mathbf{w}(x, t)\} = \sum_{r=1}^{N} \{\boldsymbol{\phi}(x)\}_r p(t)_r = [\boldsymbol{\phi}]\{\mathbf{p}(t)\}. \quad (2)$$

Here $\{\mathbf{p}(t)\}$ is a time-dependent vector of modal amplitudes, and $[\boldsymbol{\phi}]$ is a time-independent modal matrix of the N modes $\{\boldsymbol{\phi}(x)\}_r$, $r = 1, 2, \ldots, N$ of the underlying linear system, which may be obtained by solving the eigenvalue problem for undamped free vibration:

$$[\mathbf{K}_L]\{\boldsymbol{\phi}(x)\}_r = \omega_{Lr}^2[\mathbf{M}]\{\boldsymbol{\phi}(x)\}_r \quad r = 1, 2, \ldots, N. \quad (3)$$

Here ω_{Lr} is the linear natural frequency of mode "r". The number of degrees of freedom required to model the beam with reasonable accuracy can be reduced by considering only those modes $NR \ll N$ with natural frequencies in the frequency range of interest, or which are considered important in the response. In this case, the modal matrix $[\boldsymbol{\phi}]$ is reduced to dimension $(N \times NR)$ and the vector of modal amplitudes $\{\mathbf{p}(t)\}$ to dimension $(NR \times 1)$. Substituting the truncated modal expansion into the system equations of motion and pre-multiplying by $[\boldsymbol{\phi}]^{\mathrm{T}}$ yields

$$[\boldsymbol{\phi}]^{\mathrm{T}} + [\mathbf{M}][\boldsymbol{\phi}]\{\ddot{\mathbf{p}}(t)\} + \alpha[\boldsymbol{\phi}]^{\mathrm{T}}[\mathbf{M}][\boldsymbol{\phi}]\{\dot{\mathbf{p}}(t)\} + [\boldsymbol{\phi}]^{\mathrm{T}}[\mathbf{K}_L][\boldsymbol{\phi}] + \boldsymbol{\phi}^{\mathrm{T}}[\mathbf{K}_{NL}][\boldsymbol{\phi}]\{p(t)\}$$
$$= [\boldsymbol{\phi}]^{\mathrm{T}}\{\mathbf{F}(t)\}. \quad (4)$$

Upon completion of the modal transformation the new system equations of motion in modal space are

$$[\mathbf{m}]\{\ddot{\mathbf{p}}(t)\} + \alpha[\mathbf{m}]\{\dot{\mathbf{p}}(t)\} + [\mathbf{k}_L] + [\mathbf{k}_{NL}]\{\mathbf{p}(t)\} = \{\mathbf{f}(t)\}. \quad (5)$$

Here $[\mathbf{m}]$ is the modal mass matrix, $[\mathbf{k}_L]$ is the linear modal stiffness matrix and $\{f(t)\}$ is the modal force vector. It should be noted that all of the matrices in the modal equation of motion are now diagonal apart from the non-linear stiffness matrix $[\mathbf{k}_{NL}]$, which may contain cross-coupling terms and will be a function of $\{\mathbf{p}(t)\}$. The diagonalization of the linear terms occurs because of the orthogonality of the modes of the linear system.

2.2. REGRESSION ANALYSIS USING THE STATIC FINITE ELEMENT METHOD

If the beam is considered in a static sense only, with velocity and acceleration terms set to zero, and all of the geometric and material properties being time invariant, then equation (5) may be simplified as

$$[\mathbf{k}_L] + [\mathbf{k}_{NL}]\{\mathbf{p}(t)\} = \{\mathbf{f}(t)\}. \quad (6)$$

The left-hand side of equation (6) may be regarded as a stiffness restoring force, with a linear and a non-linear component. The right-hand side of equation (6) may be regarded as a statically applied load. It follows that if there exists a set of applied static loads, and corresponding displacements, then the unknown stiffness coefficients which relate the applied load to the resultant displacement may be determined using regression analysis. The set of applied loads and corresponding displacements are denoted here as "test cases", and may be solved for using a proprietary finite element software package. The following section describes a procedure for generating these test cases so that any subsequent regression analysis procedure for determining stiffness coefficients will be effective.

2.3. STRATEGY FOR GENERATING FINITE ELEMENT NON-LINEAR TEST CASES

For the case of a two-dimensional beam, all of the static non-linear finite element test cases are in the form of a non-uniform transverse distributed loading upon the beam. The transverse deflection corresponding to a particular loading may be found using proprietary finite element software. The applied load and the resultant displacement values may then be transformed into modal space, allowing them to be used as exemplars in a regression analysis.

When generating a static non-linear finite element test load case, two factors are important. The first is the spatial distribution of the load over the surface of the finite element model. Variation in the spatial characteristics of a load will result in a different level of modal participation in the load following the transformation to modal space. The second factor is the overall magnitude of the finite element loading. Varying the overall magnitude of the finite element loading will result in the inherent displacement-dependent non-linearities in the structure being exercised to a greater or lesser extent.

In the absence of *a priori* information about the exact nature of the non-linear stiffness, it must be assumed that non-linear cross-couplings are significant. Therefore, a load that is applied purely in the shape of one mode (linear force appropriation) may induce a displacement response that is a combination of more than one mode. In other words, the inclusion of non-linear cross-couplings into the non-linear model effectively extends the regression analysis problem from that of a curve fit to a multi-dimensional surface fit. For accurate identification of the modal non-linear stiffness coefficients, this multi-dimensional space needs to be adequately filled with discrete data points. Considering test load cases that are a sum of a number of weighted mode shapes has fulfilled this requirement. For a given test case

$$\{\mathbf{F}\} = a_1\{\boldsymbol{\phi}\}_1 + a_2\{\boldsymbol{\phi}\}_2 + \cdots + a_{NR}\{\boldsymbol{\phi}\}_{NR} = \sum_{r=1}^{NR} a_r\{\boldsymbol{\phi}\}_r. \tag{7}$$

Here the a_r are scalar weighting factors for each mode shape, and $\{\mathbf{F}\}$ is a vector of discrete loads in finite element nodal space. The same loads in modal space may be calculated using

$$\{(\mathbf{f})\} = [\boldsymbol{\Phi}]^{\mathrm{T}}\{\mathbf{F}\}. \tag{8}$$

The values of the coefficients a_r determine the position of the resultant modal force data point in the multi-dimensional modal force vector space. Once the relevant test load cases have been solved using proprietary non-linear finite element software, the resulting discrete data in finite element nodal space is transformed to modal space by solving

$$\{\mathbf{w}\} = [\boldsymbol{\Phi}]\{\mathbf{p}\}. \tag{9}$$

The discrete data points are included in a global data set, so that all of the possible modal cross-couplings between any two modes may be considered.

2.4. CURVE FITTING THE STIFFNESS RESTORING FORCE

Upon completion of the static non-linear test load cases, and their transformation to modal space, one has load displacement data sets in the form of vectors of applied modal forces, and corresponding vectors of modal displacements. In order to curve fit the stiffness restoring force, an ordinary polynomial approach is adopted. In each mode, the restoring force is approximated by a series of powers of modal displacements. Using the stiffness restoring force for some mode "r" as an example, gives

$$\begin{aligned} f_r(p_r, p_s, p_t) \approx \hat{f}_r(p_r, p_s, p_t) &= A_r^{(0,0,0)} + A_r^{(1,0,0)} p_r + A_r^{(2,0,0)} p_r^2 + A_r^{(1,1,0)} p_r p_s + A_r^{(0,2,0)} p_s^2 \\ &\quad + A_r^{(3,0,0)} p_r^3 + A_r^{(2,1,0)} p_r^2 p_s + \cdots \\ &= \sum_{j=0} \sum_{k=0}^{j+k+l \leqslant 3} \sum_{l=0} A_r^{(j,k,l)} p_r^j p_s^k p_t^l. \end{aligned} \tag{10}$$

Here p_s, p_t and p_u are the respective modal displacements of arbitrary modes, and $A_r^{(k,j,l)}$ are the unknown stiffness coefficients. The caret is associated with fitted rather than exact values. The value of the linear modal stiffness in each mode may be deduced from the linear mode shape/natural frequency analysis. For mode number r

$$A_r^{(1,0,0)} p_r = m_r \omega_{Lr}^2 p_r. \tag{11}$$

This term can then be moved to the left-hand side of equation (10), thus decreasing the size of the series by one:

$$f_r(p_r, p_s, p_t) - A_r^{(1,0,0)} p_r \approx \hat{f}_r(p_r, p_s, p_t)_{NL} = \sum_{j=0} \sum_{k=0}^{j+k+l \leqslant 3} \sum_{l=0} A_r^{(j,k,l)} p_r^j p_s^k p_t^l \quad j \neq 1. \tag{12}$$

The approximating function is, therefore, that of the non-linear stiffness restoring force. Note that for the majority of instances of geometric non-linearity, polynomials of up to third order have been found to be sufficient to model the non-linear response. Also in cases where the beam in question is symmetric about its centerline, then the steady-state and quadratic terms may be omitted from the regression analysis.

A least-squares backward elimination technique [23] is used to find the coefficients of the significant terms in the series for each mode, as well as to eliminate those terms from the series which do not contribute significantly to the overall response. For the purposes of illustration, a two-mode ($NR = 2$) approximation is described throughout this work, although the method has been applied in instances with more than two modes without modification.

2.5. BACKWARD ELIMINATION TECHNIQUE

Consider that there are NT sets of data vectors. Each set consists of a vector of modal displacements and a vector of modal forces. Each individual element in the modal force vector will be treated separately. Assume also that the linear modal stiffness is known and has been moved to the left-hand side of equation (10). If equation (12), which initially contains all of the possible terms which may occur in the stiffness restoring force approximating function, is evaluated at all of the NT data points, then the results may be represented in a matrix form as

$$\left\{\begin{array}{c} f_{r(1)} - A_r^{(1,0)} p_{r(1)} \\ f_{r(2)} - A_r^{(1,0)} p_{r(1)} \\ \cdot \\ \cdot \\ f_{r(NT)} - A_r^{(1,0)} p_{r(NT)} \end{array}\right\} \approx \begin{bmatrix} 1 & p_{r(1)}^2 & p_{r(1)}p_{s(1)} & p_{s(1)} & \cdot & \cdot \\ 1 & p_{r(2)}^2 & p_{r(2)}p_{s(2)} & p_{s(2)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & p_{r(NT)}^2 & p_{r(NT)}p_{s(NT)} & p_{s(NT)} & \cdot & \cdot \end{bmatrix} \times \left\{\begin{array}{c} A_r^{(0,0)} \\ A_r^{(2,0)} \\ A_r^{(1,1)} \\ A_r^{(0,2)} \\ \cdot \\ \cdot \end{array}\right\} \tag{13}$$

or using matrix shorthand notation as

$$\{\mathbf{f}_r\}_{NL} \approx \{\hat{\mathbf{f}}_r\}_{NL} = [\mathbf{D}_r]\{\mathbf{A}_r\}. \tag{14}$$

Here $\{\mathbf{f}_r\}$ is an $(NT \times 1)$ vector containing the mode r non-linear stiffness restoring force for each of the data sets, $\{\mathbf{A}_r\}$ is an $(NA \times 1)$ vector containing all of the unknown stiffness coefficients for each mode r. The matrix $[\mathbf{D}_r]$ is an $(NT \times NA)$ matrix, known as the design matrix of the fitting problem for each mode r. Each column in the design matrix corresponds to an evaluation of one model term in the series of equation (11), where the coefficients $A_r^{(k,j,l)}$ have been factored into $\{A_r\}$. Each row in the design matrix represents an evaluation using one of the NT data sets available. The subscripts in brackets refer to the test load case that the data value originated from. Note that for an accurate solution, NT is greater than NR.

Pre-multiplying equation (14) by $[\mathbf{D}_r]^{\mathrm{T}}$ yields the "normal" form of the least-squares problem:

$$[\mathbf{D}_r]^{\mathrm{T}}\{\hat{\mathbf{f}}_r\}_{NL} = [\mathbf{D}_r]^{\mathrm{T}}[\mathbf{D}_r]\{\mathbf{A}_r\}. \tag{15}$$

At this stage of the backward elimination procedure, when all possible series terms have been included in the model, there is a possibility of the normal matrix $[\mathbf{D}_r]^{\mathrm{T}}[\mathbf{D}_r]$ being singular or near to singular. In such a situation the normal equations are referred to as being ill conditioned. For this reason the singular value decomposition technique is used in the solution of the normal equations, rather than a traditional solution technique such as Gauss–Jordan elimination. When used in this context, singular value decomposition (SVD) is a technique for finding the pseudo-inverse of a rectangular matrix, and hence for solving systems of linear equations that are suspected to be ill conditioned. As such, SVD is the method of choice for solving linear least-squares problems such as the ones encountered here.

The SVD technique allows any $(NT \times NA)$ matrix $\mathbf{D}$ with $NT > NA$ to be represented as the product

$$[\mathbf{D}_r] = [\mathbf{U}][\mathbf{W}][\mathbf{V}]^{\mathrm{T}}. \tag{16}$$

Here $[\mathbf{U}]$ is an $NT \times NA$ matrix representing a column orthogonal filtered version of the components in $[\mathbf{D}_r]$, and $[\mathbf{V}]$ is an $(NA \times NA)$ orthogonal matrix. The matrix $[\mathbf{W}]$ is an $(NA \times NA)$ diagonal matrix, where all of the diagonal elements are positive, and represent the magnitudes or "singular values" of the components in $[\mathbf{U}]$. Some of the diagonal elements in [**W**] may be zero in the case of a singular matrix, or near to zero in the case of a near singular matrix. This indicates that some of the components in [**U**] do not contribute significantly to the characterization of [$\mathbf{D}_r$]. The decomposition may be "filtered" to remove the non-contributing components be setting $1/w_j$ to zero in $[\mathbf{W}]^{-1}$ when equation (14) is solved:

$$\{\mathbf{A}_r\} = [\mathbf{V}][\mathbf{W}]^{-1}[\mathbf{U}]^{\mathrm{T}}\{\hat{\mathbf{f}}_r\}_{NL}. \tag{17}$$

In the case of an ill-conditioned normal matrix $[\mathbf{D}_r]^{\mathrm{T}}[\mathbf{D}_r]$, this approach will give "sensible" values for the coefficients $\{\mathbf{A}_r\}$, although these values may not be exact in the strict algebraic sense.

Having solved for all of the coefficients $\{\mathbf{A}_r\}$, we can now find the contribution of each term in the series of equation (14) to the overall response in an r.m.s. sense. The significance of the $A_r^{(1,1)}$ term, for example, may be found by the ratio

$$\frac{\sqrt{(1/NT)\sum_{i=1}^{NT}(A_r^{(1,1)}p_{r(i)}p_{s(i)})^2}}{\sqrt{(1/NT)\sum_{i=1}^{NT}(\sum_{a=1}^{NA}A_{r(a)}^{(k,j)}p^k r^{(i)} p^j_{s(i)})}}. \tag{18}$$

Here the terms in the innermost double summation in the denominator are those terms that have not yet been eliminated from the model. If the fitted model contains more terms than are required to accurately model the system, then the non-significant terms will have a small ratio of contribution. The backward elimination technique proceeds by eliminating the term in the series that contributes the least to the overall response. Once the least contributing element has been identified, the column corresponding to this model term is removed for the design matrix [$\mathbf{D}_r$]. The curve fit procedure is then repeated with the model order decreased by one. This process of elimination is repeated until some global termination criterion is satisfied.

The cumulative goodness of fit parameter, R_T^2, is used as the criterion for terminating the backward elimination process:

$$\begin{aligned} R_T^2 &= 1 - \frac{\sqrt{\sum_{i=1}^{N}(f_{NL} - \hat{f}_{NL})^2}}{\sqrt{\sum_{i=1}^{N} f_{NL}^2}} \\ &= 1 - \frac{(\{\mathbf{f}_r\}_{NL} - [\mathbf{D}_r]\{\mathbf{A}_r\})^{\mathrm{T}}(\{\mathbf{f}_r)\}_{NL} - [\mathbf{D}_r]\{\mathbf{A}_r\})}{\{\mathbf{f}_r\}_{NL}^{\mathrm{T}}\{\mathbf{f}_r\}_{NL}}. \end{aligned} \tag{19}$$

A perfect fit will have an R_T^2 value of 1. If the cumulative goodness of fit is less than a pre-selected threshold value, then the backward elimination procedure is terminated, otherwise it will continue with a reduced model order. Upon termination of the backward elimination loop, those terms still remaining in the series, plus the last term removed, constitute the most parsimonious system model available under the present criterion. If the threshold R_T^2 value is set very close to 1, then there will be more terms retained in the series, while a lower R_T^2 threshold value (0·95 say) will yield a model with less terms (although that model may be correspondingly less accurate).

If the backward elimination procedure exhausts all of the possible model terms, then the mode in question is obviously a linear one, and no non-linear stiffness terms are required. If the backward elimination procedure terminates on the first iteration, then the model terms encapsulated in the design matrix are not sufficient to accurately model the non-linear restoring force, and more terms should be added. Once the optimum series for a particular non-linear modal restoring force has been identified, the entire backward elimination process is repeated for the next mode, until the entire multi-mode model has been identified.

The inertial and damping terms are now included to complete the governing modal equations of motion. The choice of damping model to be added at this point is arbitrary and may be non-linear or experimentally derived, although linear mass proportional damping is used here for the purposes of verification. The non-linear dynamic behavior of the beam can now be found for any form of excitation.

3. SIMULATION MODEL

In order to provide a validation of the proposed method for homogeneous, isotropic beams, both free vibration and the response of to a harmonically varying, uniformly distributed load will be considered. The geometric and material properties of the beam are summarized in Table 1. The Finite Element code ABAQUS/Standard was used to model the beam, with simply supported and fully clamped boundary conditions.

The finite element model consisted of a mesh of 10 quadratic interpolating shear deformable beam elements (ABAQUS B22). The use of such a mesh density allowed the first two symmetric modes of the beam to be modelled with reasonable accuracy [18].

Following the formulation of the beam model, a linear mode shape/natural frequency analysis was performed on the beam model using ABAQUS/Standard. For the simply supported beam, the linear natural frequencies of the first two symmetric modes were $\omega_{L1} = 22{\cdot}96$ Hz, and $\omega_{L2} = 206{\cdot}4$ Hz. For the clamped beam, the linear natural frequencies of the first two symmetric modes were $\omega_{L1} = 52{\cdot}02$ Hz, and $\omega_{L2} = 280{\cdot}6$ Hz. The non-linear dynamic modal model of the beam was then found by following the procedure of section 2, using a total of nine static non-linear test cases for a two-mode model. The values of the stiffness coefficients remaining after the regression analysis are given in Table 2 for the fully simply supported beam, and Table 3 for the fully clamped beam.

4. UNDAMPED FREE VIBRATION

A good indication of the validity of the proposed approach, for a single mode approximation of beam vibration behavior, may be obtained by considering the undamped free vibration of the initially unstressed non-linear beam. The fundamental mode expansion of the undamped beam equations of motion leads to the following Duffing-type differential equation:

$$m_1\ddot{p}_1(t) + A_1^{(1,0)}p_1(t) + A_{(1)}^{(3,0)}p_1(t)^3 = 0. \tag{20}$$

TABLE 1

Material and geometric properties of the example beam

Length, L	1 m
Thickness, a	0·01 m
Width, b	0·03 m
Mass density, ρ	7800 kg/m^3
Tensile elastic modulus, E	200×10^9 N m^2
The Poisson ratio, ν	0·3

TABLE 2

Stiffness coefficients for the two-mode fully simply supported beam model

$A_1^{(1,0)}$	$1{\cdot}217306 \times 10^4$
$A_1^{(3,0)}$	$3{\cdot}647660 \times 10^8$
$A_2^{(0,1)}$	$9{\cdot}839762 \times 10^5$
$A_2^{(0,3)}$	$3{\cdot}610174 \times 10^{10}$

TABLE 3

Stiffness coefficients for the two-mode fully clamped beam model

$A_1^{(1,0)}$	$4{\cdot}956373 \times 10^4$
$A_1^{(3,0)}$	$3{\cdot}509080 \times 10^8$
$A_1^{(2,1)}$	$-9{\cdot}074525 \times 10^8$
$A_1^{(1,2)}$	$3{\cdot}640643 \times 10^9$
$A_1^{(0,3)}$	$-2{\cdot}660977 \times 10^9$
$A_2^{(0,1)}$	$1{\cdot}599697 \times 10^6$
$A_1^{(0,3)}$	$2{\cdot}779785 \times 10^{10}$
$A_1^{(1,2)}$	$-8{\cdot}119428 \times 10^9$
$A_1^{(2,1)}$	$3{\cdot}645975 \times 10^9$
$A_1^{(3,0)}$	$-2{\cdot}975476 \times 10^8$

The values of $A_1^{(1,0)}$ and $A_1^{(3,0)}$ are given in Tables 2 and 3 for the fully simply supported beam and fully clamped beam respectively. This equation may be solved in terms of the ratio of the non-linear natural frequency, ω_{NL}, corresponding to vibration at large amplitudes, to the linear natural frequency, ω_L, corresponding to vibration at very small amplitudes. The exact solution for the frequency ratio was obtained by using the elliptic integral approach [24] as used by Woinowsky-Krieger [3]. The elliptic integral approach for free vibration is applicable to a single-mode approximation of the beam vibration, where non-linear cross-couplings are assumed to be negligible and thus is only an approximation in the case of the fully clamped beam. Table 4 compares the frequency ratio of the fundamental mode of the simply supported beam at various amplitudes calculated using differing methods by various researchers. The values are given at various normalized half-amplitudes, where W_0 is the half-amplitude of the beam center, and $\hat{\imath}$ is the radius of gyration of the beam cross-section. A very brief description of the methods used by some of these investigators is given in the introduction. It can be seen from Table 4 that the results obtained using the present method are very similar to the benchmark values of Woinowsky–Krieger [3], even though the stiffness coefficients are obtained using the combined modal/finite element approach rather than an analytical approach. Table 5 compares the frequency ratio of the fundamental mode of the clamped beam at various amplitudes calculated using differing methods by various researchers. Again the results are comparable with those of other researchers.

TABLE 4

Natural frequency ratios of the simply supported beam from various formulations

W_0/ξ	1·0	3·0	5·0
Present work	1·0896	1·6246	2·3480
Woinowsky–Krieger [3]	1·0892	1·6257	2·3501
Mei [15, 16]	1·0613	1·4617	2·0378
Raju *et al.* [4]	1·0897	1·6394	2·3848
Lewandowski [5]			
Sarma and Varadan [13]	1·1180	1·8028	2·6926
Mei and Decha-Umphai [17] Without ELDI*	1·0888	1·6022	2·2544
Mei and Decha-Umphai [17] with ELDI*	1·0613	1·4617	2·0378
Singh *et al.* [14]	1·0892	1·6257	2·3502
Shi *et al.* [19]	1·0892	1·6258	2·3506

*Effects of longitudinal deformation and inertia.

TABLE 5

Natural frequency ratios of the clamped beam from various formulations

W_0/ξ	1·0	3·0	5·0
Present Work	1·0206	1·1967	1·4220
Shi and Mei [18]	1·0221	1·1842	—
Singh *et al.* [4]	1·0221	1·1825	1·4474
Lewandowski [5]	1·0222	1·1833	1·4476
Evensen [25]	1·0222	1·1853	1·4577
Sarma and Varadan [13]	1·0295	1·2377	1·5659
Azrar *et al.* [26]	1·0222	1·1831	1·4488

5. DAMPED FORCED VIBRATIONS—HARMONIC EXCITATION

In this section, the ability of the reduced two-mode model to yield accurate responses of the beam to a harmonically varying distributed load is considered. As a means of illustrating the method, the response to a number of load time histories were analyzed using time domain numerical integration. For all of the cases the sinusoidal load time history was discretized such that there were at least 60 sampling points per cycle of excitation. The same time history was used for both the reduced modal method, and the direct finite element approach. This meant that the third and fifth harmonic components of the response could be modelled with reasonable accuracy. In all cases, a mass proportional damping factor of 1·0 and a two symmetric mode solution was used. For a selection of cases, the time domain response of the modal model is compared with the direct integration finite element method. The fourth-order Runge–Kutta integration algorithm was used for the modal model, while the implicit integration routine used in ABAQUS/Standard was a Hilber Hughes Taylor integration operator [27]. In all cases, a steady state is allowed to develop before the properties of the vibration are investigated.

5.1. SIMPLY SUPPORTED BEAM

A loading that is uniformly distributed along the beam and sinusoidally varying in time is considered for the simply supported beam. The harmonic excitation was of the form

$$F(t) = 500 \sin(2\pi f_e t). \tag{21}$$

Here F is the uniformly distributed load, measured in N/m along the beam, and f_e is the frequency of excitation. The excitation frequency was varied from 0 to 4·5 times the fundamental linear natural frequency. The variation in the r.m.s. response level with respect to excitation frequency is shown in Figure 1. Note the presence of bifurcation or "jump" phenomena, where the beam has more than one stable vibration state, depending on the initial conditions of the vibration. This bifurcation phenomena is characteristic of non-linear systems with hardening stiffness non-linearities that are subjected to harmonic excitations. The response of the simply supported beam is considered in detail at four excitation frequencies. These are the points (1)–(4) shown in Figure 1.

Figure 2(a,b,c) shows the displacement time history, phase plot, and displacement power spectral density, respectively, of the simply supported beam at $f_e = 10$ Hz (approximately half of the fundamental linear natural frequency). As the beam only vibrates in its symmetrical modes when subjected to uniform normal pressure excitation, the middle of the beam is taken as the reference for the displacement response. The modal and complete finite element approaches are compared in the phase plot and the time history, and it may be seen that the response as calculated by the two methods is nearly identical. The frequency response is composed of both odd and even super-harmonic frequency components, as well as a component at the excitation frequency. The presence of even frequency components in such an instance is known as time symmetry breaking non-linearity.

Figures 3–5 correspond to the responses of the beam at 22·5, 56·25 and 90 Hz respectively. In these figures the response is dominated by the fundamental frequency component, especially in Figure 5 where the response is pseudo-harmonic. The response is only mildly non-linear at this frequency as the displacement amplitude is small, and therefore there is less axial stretching of the beam.

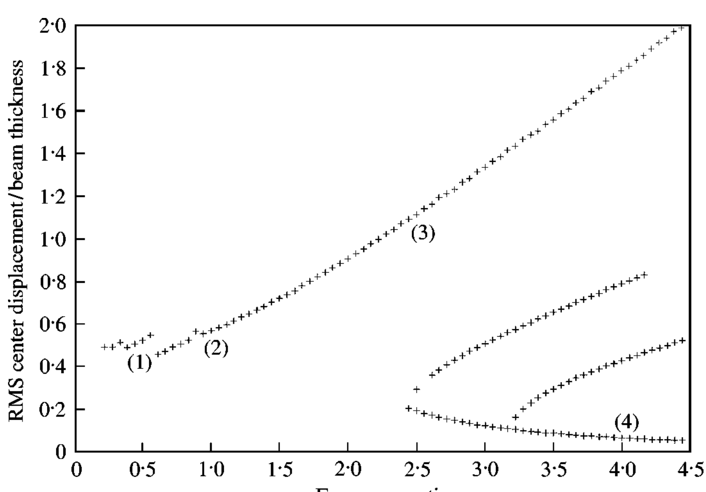

Figure 1. Root mean square frequency response of the fully simply supported beam.

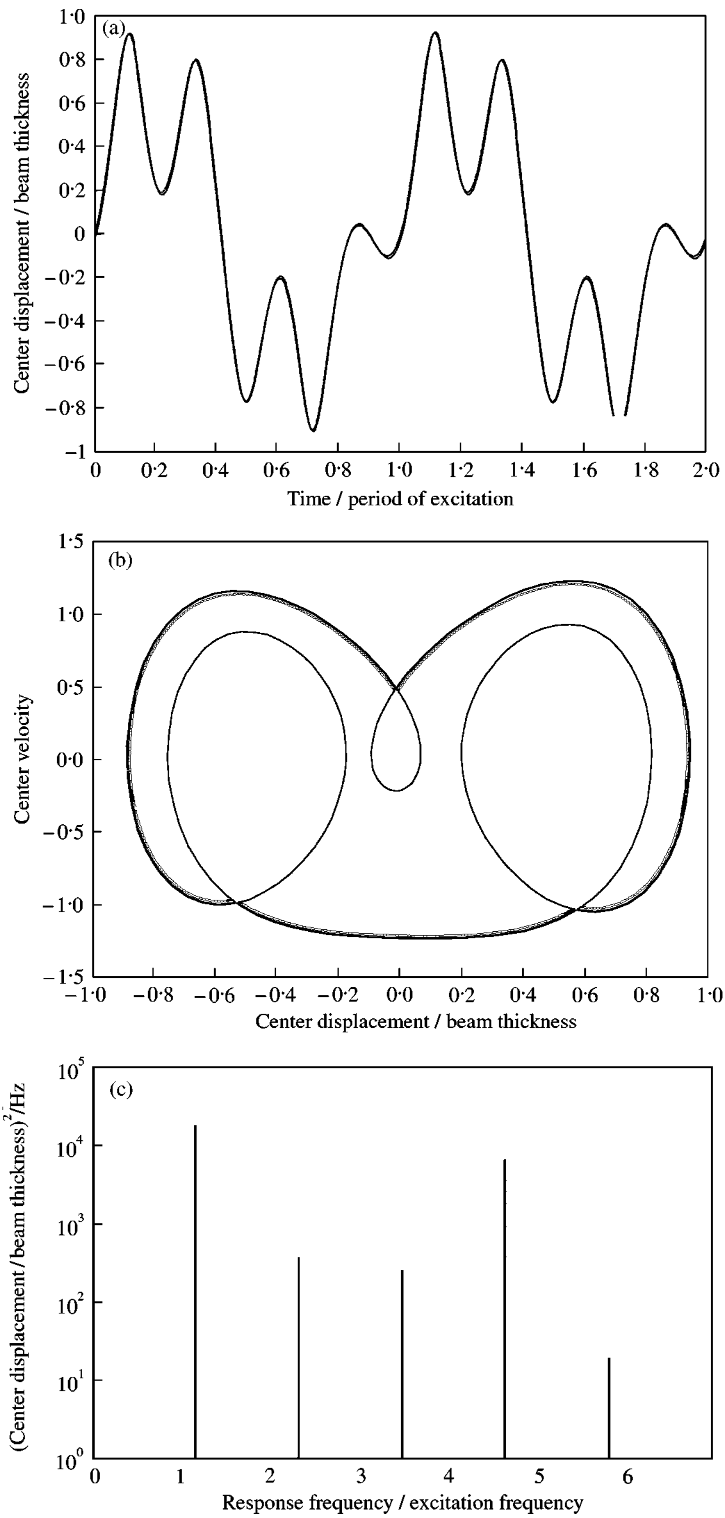

Figure 2. (a) Displacement time history of the fully simply supported beam response at $f_e \approx 0{\cdot}5 f_1$ (10 Hz). (b) Phase plot of the fully simply supported beam response at $f_e \approx 0{\cdot}5 f_1$ (10 Hz). (c) Autopower spectral density of the fully simply supported beam response at $f_e \approx 0{\cdot}5 f_1$ (10 Hz): ——, Proposed modal method; ---------, Finite element method.

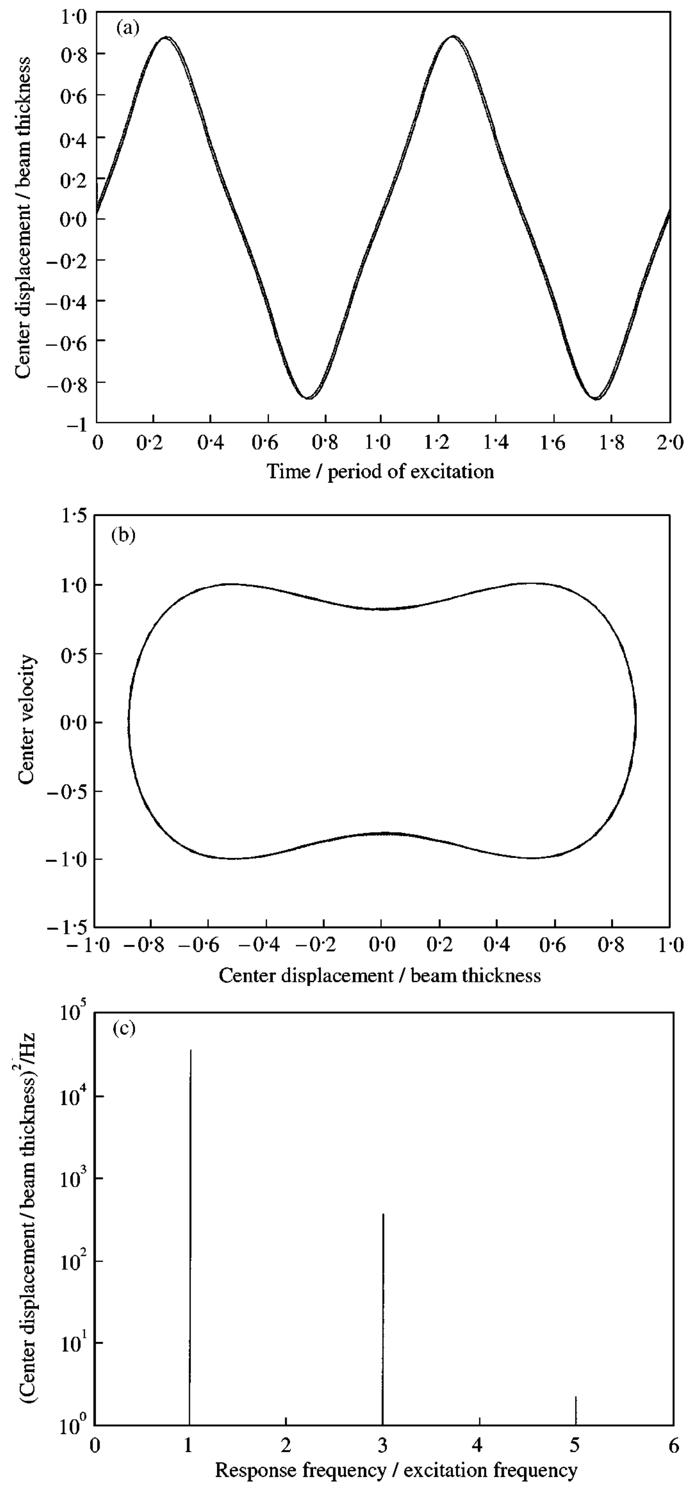

Figure 3. (a) Displacement time history of the fully simply supported beam response at $f_e \approx f_1$ (22·5 Hz). (b) Phase plot of the fully simply supported beam response at $f_e \approx f_1$ (22·5 Hz). (c) Autopower spectral density of the fully simply supported beam response at $f_e \approx f_1$ (22·5 Hz): ——, Proposed modal method; ---------, Finite element method.

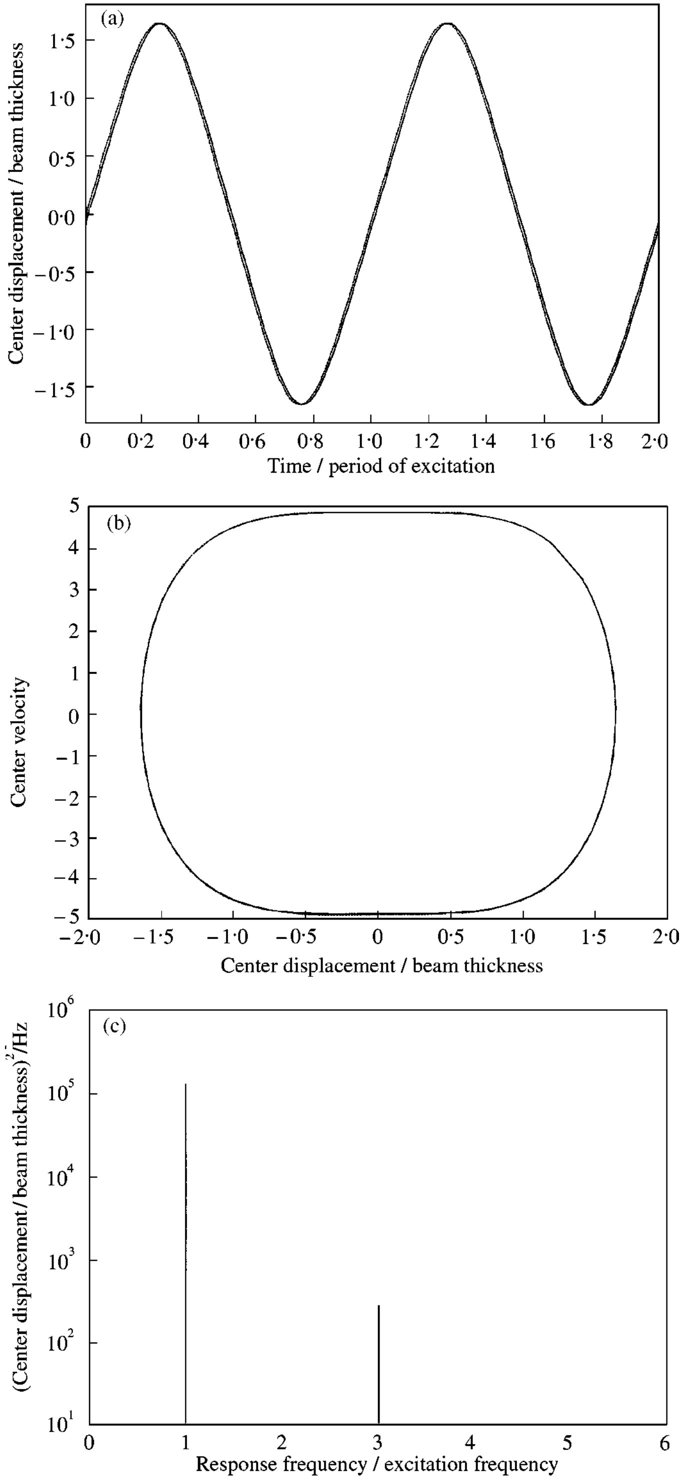

Figure 4. (a) Displacement time history of the fully simply supported beam response at $f_e \approx 2{\cdot}5f_1$ (56·25 Hz). (b) Phase plot of the fully simply supported beam response at $f_e \approx 2{\cdot}5f_1$ (26·25 Hz). (c) Autopower spectral density of the fully simply supported beam response at $f_e \approx 2{\cdot}5f_1$ (56·25 Hz). ——, Proposed modal method; ---------, Finite element method.

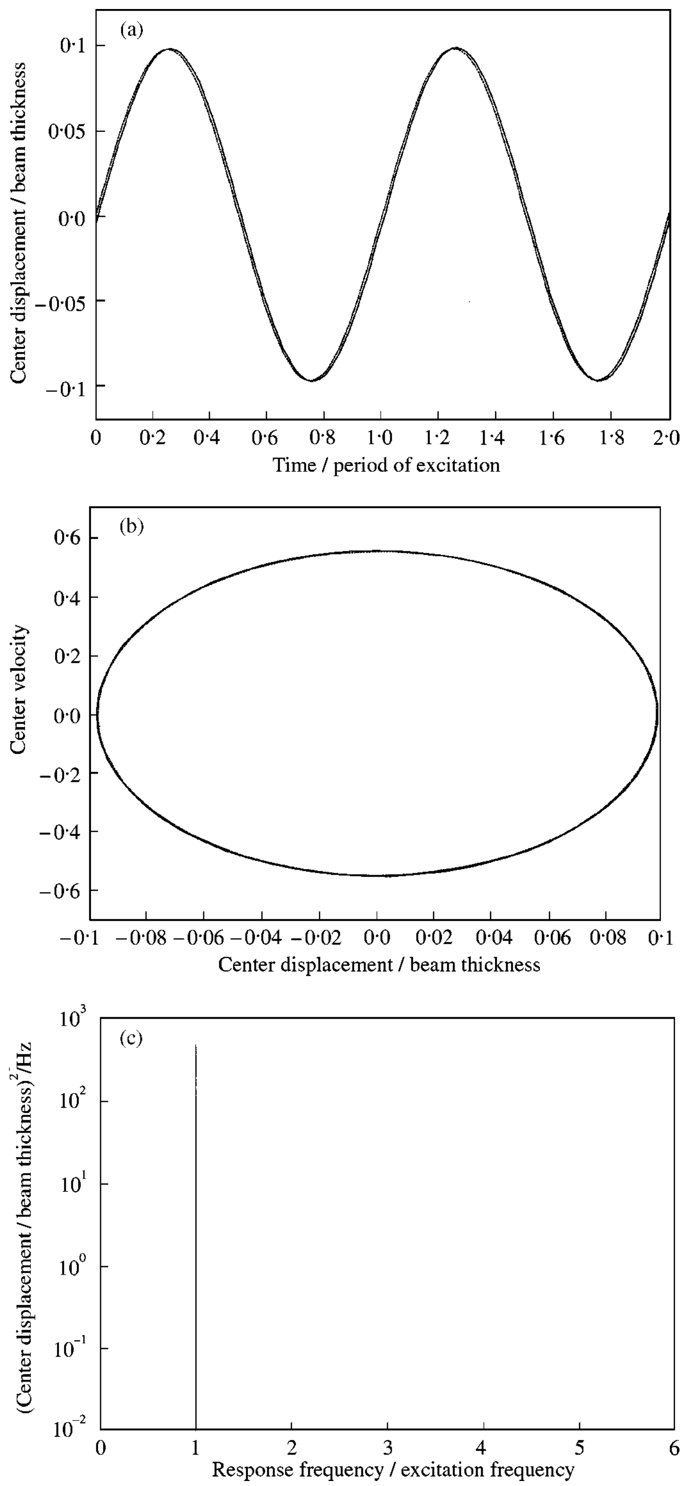

Figure 5. (a) Displacement time history of the fully simply supported beam response at $f_e \approx 4f_1$ (90 Hz). (b) Phase plot of the fully simply supported beam response at $f_e \approx 4f_1$ (90 Hz). (c). Autopower spectral density of the fully simply supported beam response at $f_e \approx 4f_1$ (90 Hz). ——, Proposed modal method; ---------, Finite element method.

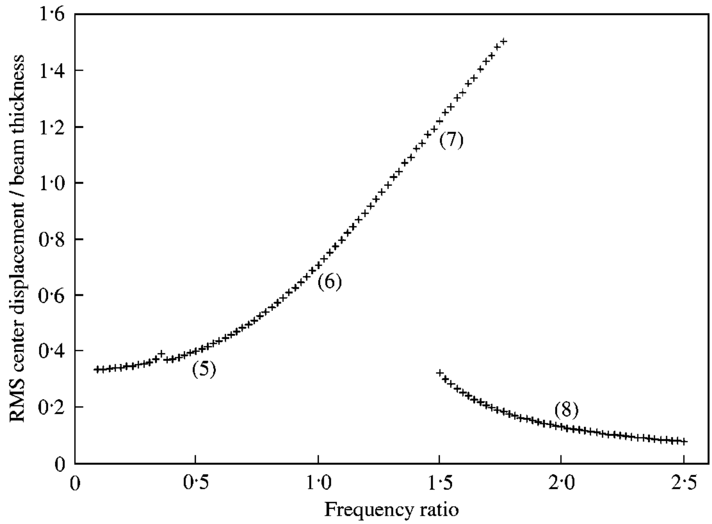

Figure 6. Root mean square frequency response of the fully clamped beam.

5.2. CLAMPED BEAM

A loading that is uniformly distributed along the beam and sinusoidally varying in time is considered for the fully clamped beam. The harmonic excitation was of the form

$$F(t) = 1000\sin(2\pi f_e t). \tag{22}$$

Here F is the uniformly distributed load, measured in N/m along the beam, and f_e is the frequency of excitation. The excitation frequency was varied from 0 to 2·5 times the fundamental linear natural frequency. The variation in the r.m.s. response level with respect to the excitation frequency is shown in Figure 6. Again the beam exhibits bifurcation behavior in the frequency range considered.

Figures 7–10 correspond to the response at the points (5)–(8) shown in Figure 6. The excitation frequencies are 18·75, 52·5, 78·75, and 105 Hz respectively. At the lowest frequency considered (18·75 Hz), the response is composed of the fundamental component plus significant superharmonic components of order 3, 5, etc. At the highest frequency considered (105 Hz), the response is quasi-harmonic due to the lower level of displacement amplitude. In all cases there is good agreement between the proposed modal method and the finite element method.

5.3. OTHER COMMENTS

The principal advantage of the proposed method is that the flexibility of modelling arbitrary structures using a proprietary finite element code is retained, while at the same time considerable computational expense may be saved in the use of a reduced modal co-ordinate system. As an example, for an analysis involving 250 000 non-linear time steps, the solution time is over 360 times less for the proposed modal method when compared to the direct finite element method. The time-savings associated with the adoption of the

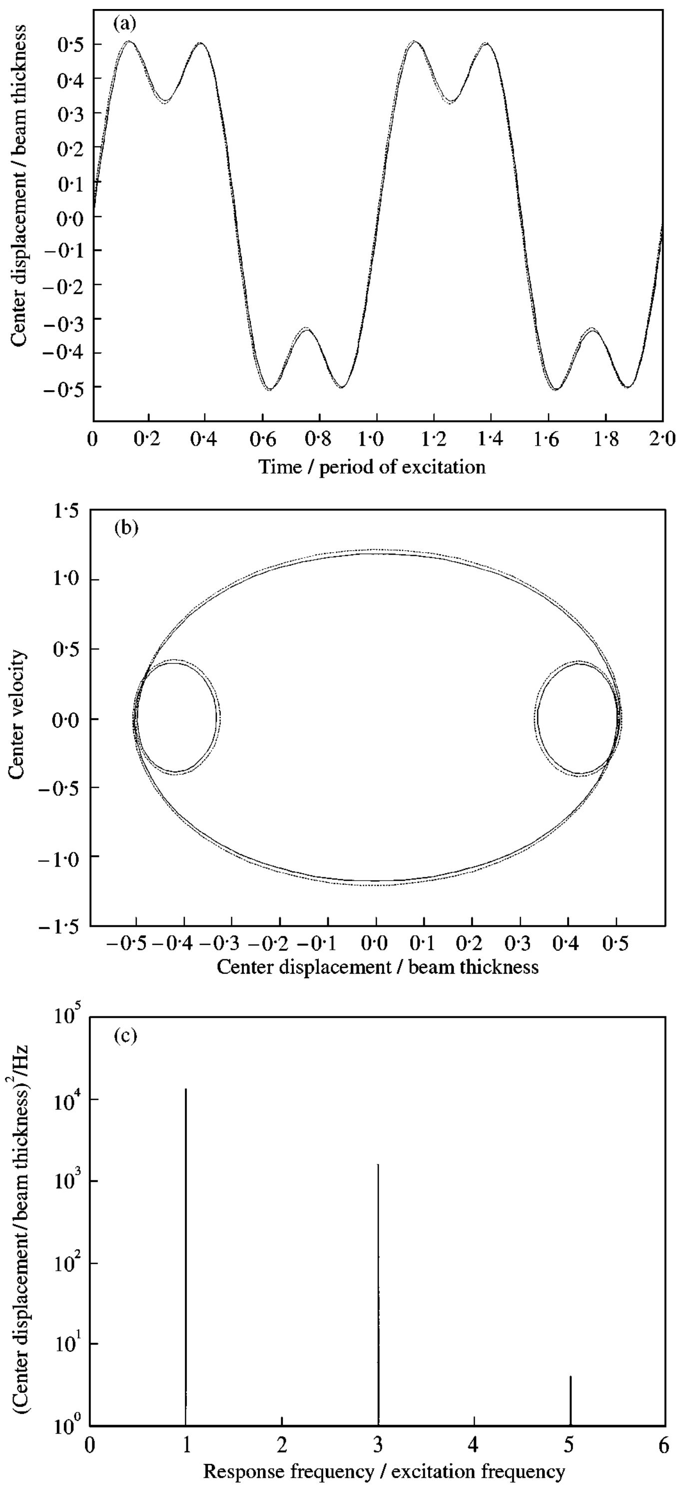

Figure 7. (a) Displacement time history of the fully clamped beam response at $f_e \approx 0{\cdot}5 f_1$ (18·75 Hz). (b) Phase plot of the fully clamped beam response at $f_e \approx 0{\cdot}5 f_1$ (18·75 Hz). (c). Autopower spectral density of the fully clamped beam response at $f_e \approx 0{\cdot}5 f_1$ (18·75 Hz). ——, Proposed modal method; ---------, Finite element method.

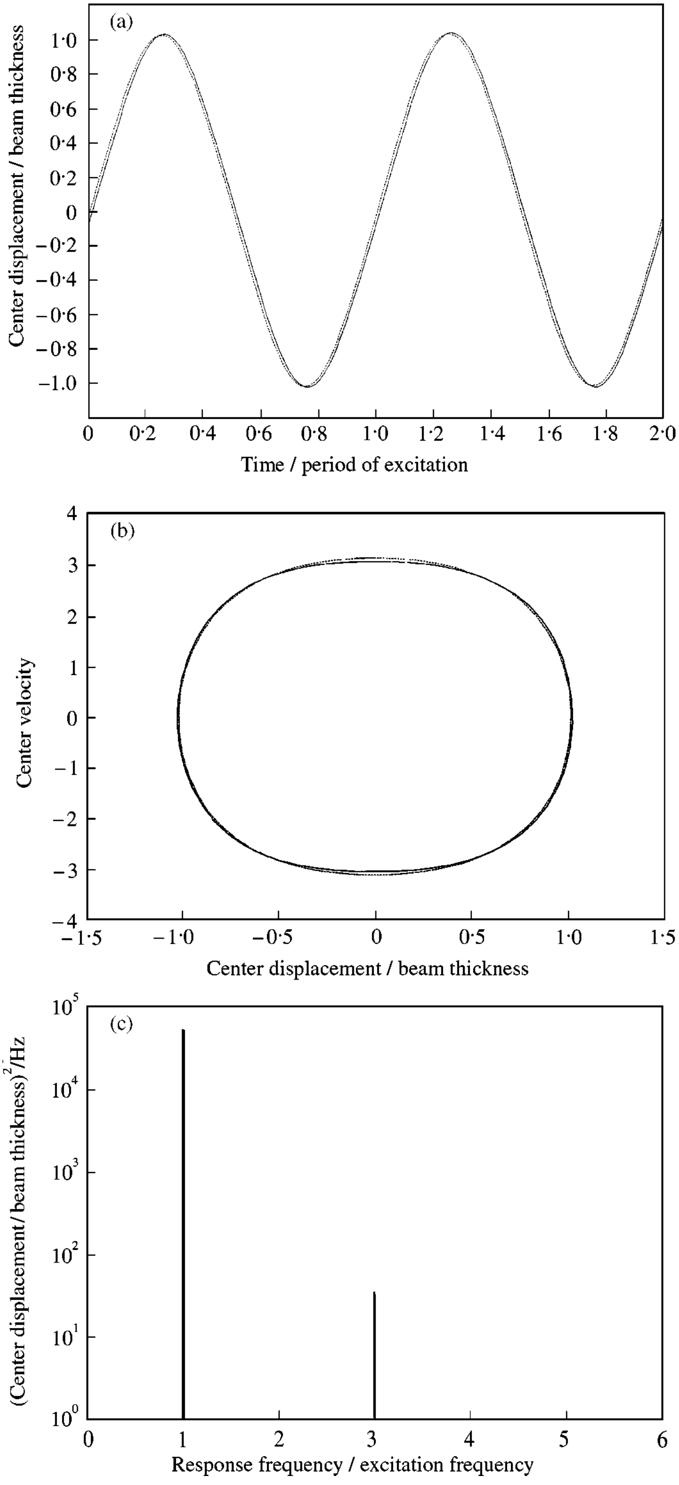

Figure 8. (a) Displacement time history of the fully clamped beam at $f_e \approx f_1$ (52·5 Hz). (b) Phase plot of the fully clamped beam response at $f_e \approx f_1$ (52·5 Hz). (c) Autopower spectral density of the fully clamped beam response at $f_e \approx f_1$ (52·5 Hz). ——, Proposed modal method; ---------, Finite element method.

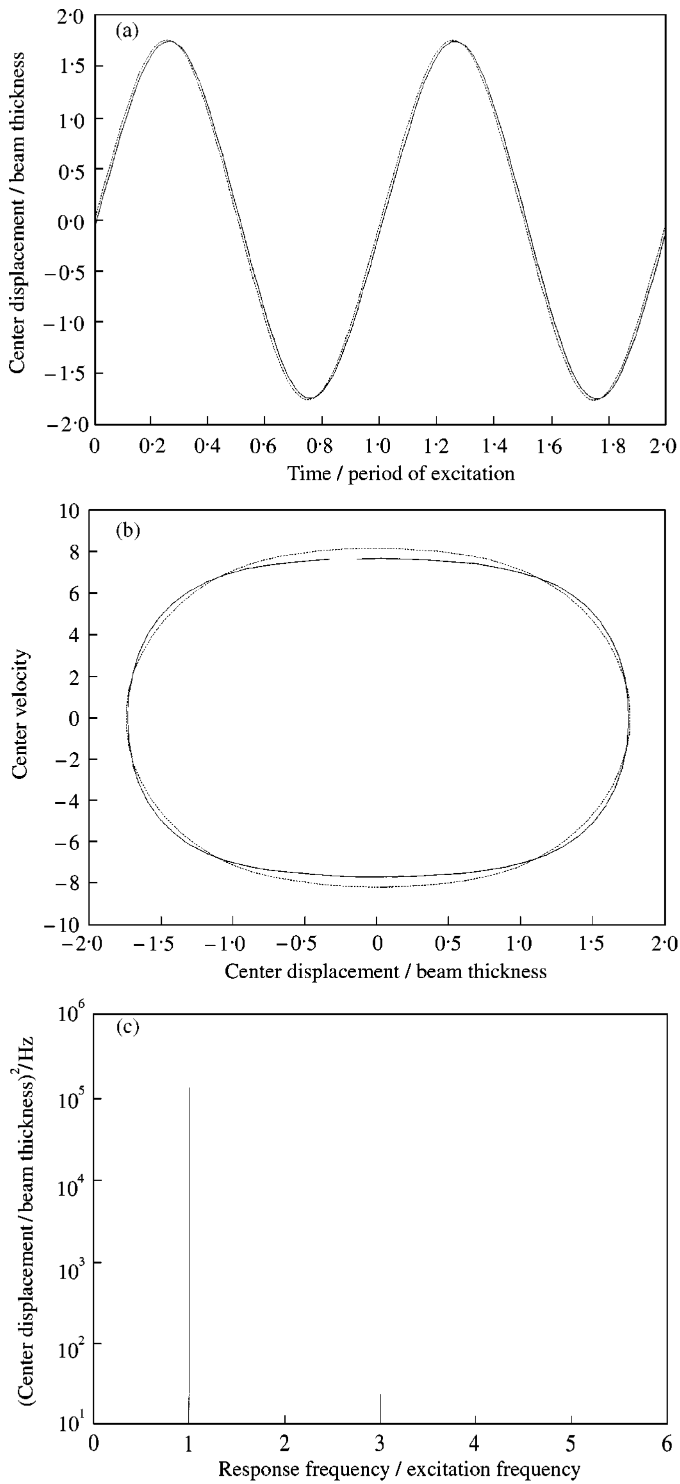

Figure 9. (a) Displacement time history of the fully clamped beam response at $f_e \approx 1{\cdot}5f_1$ (78·75 Hz). (b) Phase plot of the fully clamped beam response at $f_e \approx 1{\cdot}5f_1$ (78·75 Hz). (c) Autopower spectral density of the fully clamped beam response at $f_e \approx 1{\cdot}5f_1$ (78·75 Hz). ——, Proposed modal method; ---------, Finite element method.

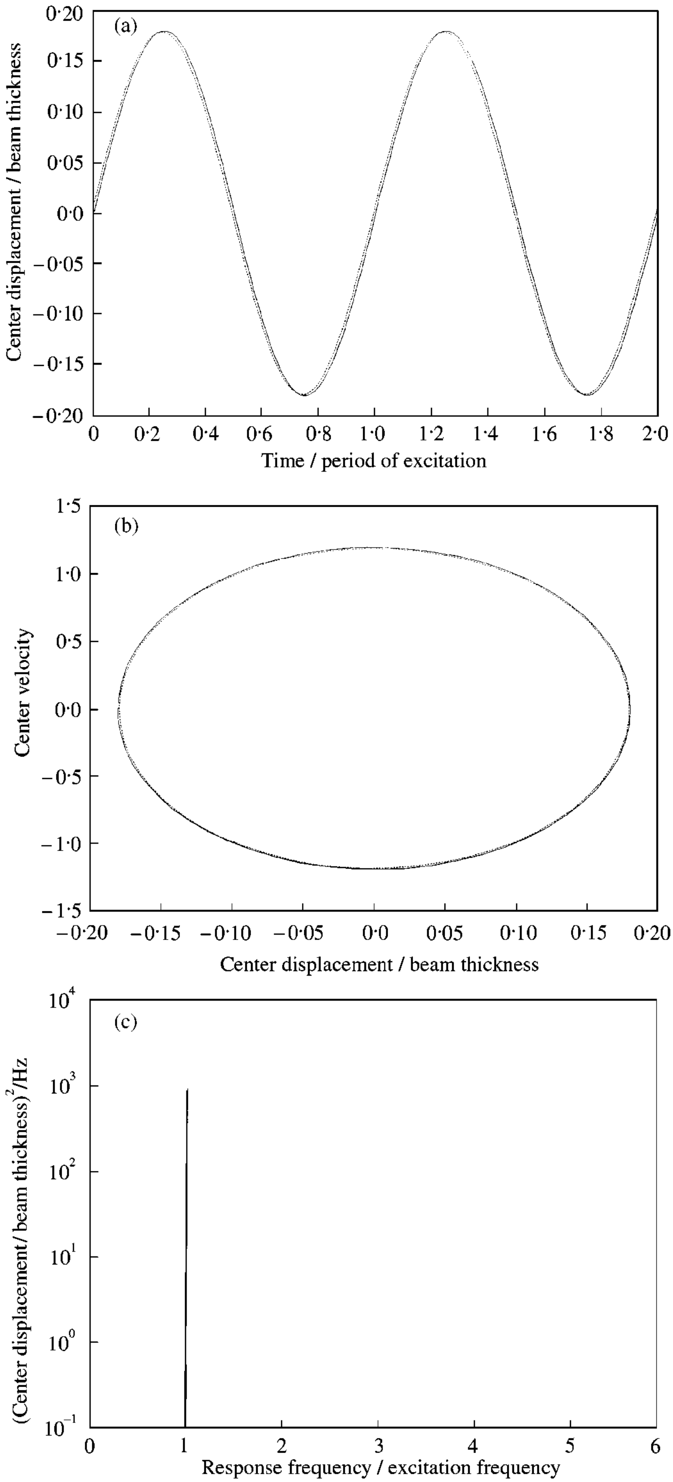

Figure 10. (a) Displacement time history of the fully clamped beam response at $f_e \approx 2f_1$ (105 Hz). (b) Phase plot of the fully clamped beam response at $f_e \approx 2f_1$ (105 Hz). (c) Autopower spectral density of the fully clamped beam response at $f_e \approx 2f_1$ (105 Hz). ——, Proposed modal method; ---------, Finite element method.

reduced modal model will become far more dramatic for more complex problems, where typically many more finite element degrees of freedom are required.

While the method described in section 2 has only dealt with polynomial-type non-linearities of a fixed order, the method may be easily extended to enable the consideration of other types of stiffness non-linearities. The only non-linear terms considered in the present work were cubic polynomials, and it was found that using only these non-linear terms resulted in the most accurate model in any case. For other types of non-linearity than stiffening non-linearities, such as clearance non-linearities, it would be more appropriate to consider using non-polynomial terms in the modal model.

It should also be noted that the proposed method is an approximate method based upon the finite element method, which is in itself an approximation of the "true" non-linear beam vibration. The overall quality of the results given by employing this method will be dependent upon the accuracy of the original static finite element model used to model the beam, as well as the degree of truncation in the reduced modal model.

6. CONCLUSION

In this work the finite element based modal approach of Maymon [9] is extended to consider multi-modal beam response. The output from a series of static finite element "test cases" is transformed into modal co-ordinates using the mode shapes of the underlying linear system. Regression analysis is then performed in order to extract the nonlinear stiffness coefficients in the modal co-ordinate system. Both direct non-linear terms and non-linear cross-coupling terms may be included in the model. The beam problem can then be solved for any force–time history in the reduced degree of freedom modal system. The method is an approximation, and is "simplified" from the point of view that considerations of finite element formulation and solution are handled by the proprietary finite element code, and are not dealt with explicitly by the method. However, the model is genuinely non-linear, with no linearization attempted.

The proposed method is applied to the case of a homogeneous isotropic beam, with fully simply supported and fully clamped boundary conditions. For the single mode, free vibration case, the results show good agreement with those of the literature. For the case of steady-state harmonic excitation with fully simply supported and fully clamped boundary conditions, the results compare well with the standard direct integration finite element approach, with a significant saving in computational expense. Future work will address random vibration, modal truncation, and the application of the approach to stress as well as displacement analysis.

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APPENDIX A: NOMENCLATURE

t	time
x	spatial co-ordinate along the axis of the beam
$[\mathbf{M}]$	assembled finite element mass matrix
$\{\mathbf{w}\}$	vector of assembled finite element transverse displacements
α	coefficient of linear mass proportional damping
$[\mathbf{K}_L]$	assembled linear finite element stiffness matrix
$[\mathbf{K}_{NL}]$	assembled non-linear finite element stiffness matrix
$\{\mathbf{F}\}$	assembled finite element force vector
r	mode number
N	number of degrees of freedom of the assembled finite element model
NR	number of degrees of freedom of the reduced modal model
$\{\boldsymbol{\phi}\}_r$	displacement mode shape vector for mode r
$\mathbf{p}_r$	displacement modal amplitude coefficient for mode r
$[\boldsymbol{\phi}]$	displacement mode shape matrix
$\{\mathbf{p}\}$	vector of displacement modal amplitudes
$[\mathbf{m}]$	modal mass matrix
$[\mathbf{k}_L]$	linear modal stiffness matrix
$[\mathbf{k}_{NL}]$	non-linear modal stiffness matrix
$\{\mathbf{f}\}$	modal force vector
a_r	component of mode shape "r" in the finite element test case load
$A_r^{(j,k,l)}$	modal non-linearity coefficient, pertaining to mode "r" involving the coupling of arbitrary modes j, k, and l
ω_{Lr}	linear natural frequency of mode "r"
$\{\mathbf{f}_r\}$	vector of test case modal forces for mode "r"
$\{\mathbf{A}_r\}$	vector of stiffness coefficients in modal space for mode "r"
NT	the total number of static non-linear test cases
NA	the total number of non-linear terms remaining in the model for a particular backward elimination iteration
$[\mathbf{D}]$	the design matrix of the linear least-squares curve fitting procedure
$[\mathbf{U}]$	an orthogonal matrix obtained by the singular value decomposition technique
$[\mathbf{W}]$	a matrix containing the singular values of the design matrix $[\mathbf{D}]$
$[\mathbf{V}]$	an orthogonal matrix obtained by the singular value decomposition technique
R_T^2	cumulative goodness of fit parameter
ξ	radius of gyration of the beam cross-section for bending
W_0	amplitude of the beam at the instant of maximum deflection
f_e	excitation frequency for harmonic excitation